HOSSAM GHANEM

(20) 3.6 The Chain Rule

Example 1
7 June 17, 1993

Suppose f is a function given by $f(x) = (1+5x)^{10}$, $x \in \mathcal{R}$

. Find $\lim\limits_{x\to 0} \dfrac{f(x)-1}{x}$

Solution

$f(x) = (1+5x)^{10}$ $\qquad f(0) = 1$

$f'(x) = 10(1+5x)^9 \cdot 5 = 50(1+5x)^9$

$f'(0) = 50(1+0)^9 = 50$

$L = \lim\limits_{x\to 0} \dfrac{f(x)-1}{x} = \lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = f'(0) = 50$

Example 2
52 April 9, 2009 A

Evaluate the following limit, if it exists $\lim\limits_{x\to 1} \dfrac{x^{\frac{2}{3}}-1}{x-1}$

Solution

Let $f(x) = x^{\frac{2}{3}}$ $\qquad f(1) = 1$ $\qquad f'(x) = \dfrac{2}{3}x^{\frac{-1}{3}}$

$\lim\limits_{x\to 1} \dfrac{x^{\frac{2}{3}}-1}{x-1} = \lim\limits_{x\to 1} \dfrac{f(x)-f(1)}{x-1} = f'(1) = \dfrac{2}{3}(1)^{\frac{-1}{3}} = \dfrac{2}{3}$

Example 3
8 August 28, 1993

Evaluate $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{4\sin^4\left(\frac{x}{2}\right)-1}{x-\frac{\pi}{2}}$

Solution

Let $f(x) = 4\sin^4\left(\dfrac{x}{2}\right)$ $\Rightarrow \therefore f\left(\dfrac{\pi}{2}\right) = 4\sin^4\left(\dfrac{\pi}{4}\right) = 4\left(\dfrac{1}{\sqrt{2}}\right)^4 = 4\cdot\dfrac{1}{4} = 1$

$L = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{4\sin^4\left(\frac{x}{2}\right)-1}{x-\frac{\pi}{2}} = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{f(x)-f\left(\frac{\pi}{2}\right)}{x-\frac{\pi}{2}} = f'\left(\dfrac{\pi}{4}\right)$

$f'(x) = 16\sin^3\frac{x}{2}\cdot\cos\frac{x}{2}\cdot\frac{1}{2}$

$L = f'\left(\dfrac{\pi}{4}\right) = 8\sin^3\dfrac{\pi}{4}\cdot\cos\dfrac{\pi}{4} = 8\left(\dfrac{1}{\sqrt{2}}\right)^3\cdot\dfrac{1}{\sqrt{2}} = 8\cdot 4 = 32$

Example 4
59 9 July 2011

[2 + 2 pts.] : Consider the function $g(x) = \left(\sin(\pi x) + \sqrt{x} \right)^3$
(a) Find $g'(x)$.
(b) Find an equation for the tangent line to the curve $y = g(x)$ at $x = 1$

Solution

$$g'(x) = 3\left(\sin(\pi x) + \sqrt{x} \right)^2 \left(\pi \cos(\pi x) + \frac{1}{2\sqrt{x}} \right)$$

$$g'(1) = 3\left(\sin(\pi) + \sqrt{1} \right)^2 \left(\pi \cos(\pi) + \frac{1}{2\sqrt{1}} \right) = 3(0 + 1)^2 \left(\pi(-1) + \frac{1}{2} \right) = 3\left(\frac{1}{2} - \pi \right) = \frac{3}{2}(1 - 2\pi)$$

$$g(1) = \left(\sin(\pi) + \sqrt{1} \right)^3 = (0 + 1)^3 = 1$$

$$m = \frac{3}{2}(1 - 2\pi) \qquad p(1, 1)$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{3}{2}(1 - 2\pi)(x - 1)$$

Example 5
44 November 9, 2006

Find $\frac{dy}{dx}$ where $y = [\cos(x^2 - 1)]^5$

Solution

$$\frac{dy}{dx} = 5[\cos(x^2 - 1)]^4 \cdot [-\sin(x^2 - 1)] \cdot 2x$$

Example 6
6 April 8, 1993

Find y' if $y = \cos^3\left(\sec^2\left(\sqrt{x^2 - 3x + 1} \right) \right)$

Solution

$$y' = 3\cos^2\left[\sec^2\left(\sqrt{x^2 - 3x + 1} \right) \right] \cdot \left(-\sin\left[\sec^2\left(\sqrt{x^2 - 3x + 1} \right) \right] \right) \cdot 2\sec\sqrt{x^2 - 3x + 1}$$
$$\cdot \sec\sqrt{x^2 - 3x + 1} \cdot \tan\sqrt{x^2 - 3x + 1} \cdot \frac{2x - 3}{2\sqrt{x^2 - 3x + 1}}$$

Example 7
43 June 28, 2008

Let $f(x) = (x^2 - 1)\sec\left(\sqrt{x - 1}\right) + \tan(x^3 - 1)$. Find $f'(x)$.

Solution

$$f'(x) = 2x\sec\left(\sqrt{x - 1}\right) + (x^2 - 1)\sec\left(\sqrt{x - 1}\right)\tan\sqrt{x - 1} \cdot \frac{1}{2\sqrt{x - 1}} + \sec^2(x^3 - 1) \cdot 3x^2$$

Example 8
39 July 3, 2004

Find $\frac{dy}{dx}$ where $y = \frac{x \sin x}{1 + \cot x}$

Solution

$$\frac{dy}{dx} = \frac{(1 + \cot x)(\sin x + x\cos x) - x\sin x\,(-\csc^2 x)}{(1 + \cot x)^2}$$

Example 9

Let f be function such that $f'(x)=\frac{1}{x}$ find

$$\frac{d}{dx}\left(f(\sec x+\tan x)\right)$$

Solution

$$\frac{d}{dx}\left(f(\sec x+\tan x)\right)=f'(\sec x+\tan x)\cdot\frac{d}{dx}(\sec x+\tan x)$$

$$=\frac{1}{\sec x+\tan x}\cdot(\sec x\tan x+\sec^2 x)=\frac{\sec x\,(\tan x+\sec x)}{\sec x+\tan x}=\sec x$$

Example 10

If $f'(x)=\sin(x^2)$ and $y=f\left(\frac{2x-1}{x+1}\right)$ find $\frac{dy}{dx}$

Solution

$$\frac{dy}{dx}=f'\left(\frac{2x-1}{x+1}\right)\cdot\frac{d}{dx}\left(\frac{2x-1}{x+1}\right)=\sin\left(\frac{2x-1}{x+1}\right)^2\cdot\frac{(x+1)\cdot 2-(2x-1)}{(x+1)^2}$$

Example 11

36 April 19, 2003 A

If $y=\left(\frac{u-1}{u+1}\right)^3$ and $u=\sqrt[3]{2-3t-t^2}$, Find $\frac{dy}{dt}$ at $t=2$

Solution

$$\frac{dy}{dt}=\frac{dy}{du}\cdot\frac{du}{dt}\qquad at\ t=2\qquad \Rightarrow u=\sqrt[3]{2-3t-t^2}=\sqrt[3]{-8}=-2$$

$$\frac{dy}{dt}=3\left(\frac{u-1}{u+1}\right)^2\cdot\frac{(u+1)-(u-1)}{(u+1)^2}\cdot\frac{1}{3}(2-3t-t^2)^{\frac{2}{3}}\cdot(-3-2t)$$

$$=3\left(\frac{1}{3}\right)^2\cdot\frac{-1-(-3)}{1}\cdot\frac{1}{3}(-8)^{\frac{2}{3}}\cdot(-3-4)$$

$$=\frac{1}{3}\cdot\frac{2}{1}\cdot\frac{1}{3}\cdot 4\cdot(-7)=\frac{-56}{9}$$

Example 12

42 March 29, 2006

Find $\frac{dy}{dx}$ If $y=(u^2+u+1)^2$ and $u=\sqrt[3]{x\sec x+1}$

Solution

$$y=(u^2+u+1)^2\qquad u=(x\sec x+1)^{\frac{1}{3}}$$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=2(u^2+u+1)\cdot(2u+1)\cdot\frac{1}{3}(x\sec x+1)^{\frac{-2}{3}}\cdot(\sec x+x\sec x\tan x)$$

Example 13

25 December 10, 2000

Find $f'(\theta)$ if $f(\theta)=\theta^2.\sec\theta-\frac{4}{\left(1+\sqrt[4]{\theta}\right)^3}$

Solution

$$f(\theta)=\theta^2.\sec\theta-\frac{4}{\left(1+\sqrt[4]{\theta}\right)^3}=\theta^2.\sec\theta-4\left(1+\theta^{\frac{1}{4}}\right)^{-3}$$

$$f'(\theta)=2\theta\sec\theta+\theta^2.\sec\theta\tan\theta-4(-3)\left(1+\theta^{\frac{1}{4}}\right)^{-4}\cdot\frac{1}{4}\theta^{\frac{-3}{4}}$$

Homework

1	Evaluate $\lim_{x\to 1}\dfrac{(4t-3t^2)^{50}-1}{t-1}$	
2	Given that $f'(2)=44$ find $\lim_{x\to 0}\dfrac{f(x)-f(2)}{x^2-4}$	1 January 1990
3	Evaluate $\lim_{x\to 0}\dfrac{\sqrt[3]{x+8}-2}{x}$	
4	Evaluate $\lim_{x\to 0}\dfrac{(x^2+x-1)^{50}-1}{x}$	
5	Find $f'(x)$, if $f(x)=x^3\tan 2x+\sec(x^2+4)$	35 October 31, 2002 A
6	Find $f'(x)$ where $f(x)=\dfrac{\csc^2 x}{5+\csc 3x}$	
7	Find $f'(x)$ where $f(x)=\sin^3\left(\dfrac{x}{\sqrt{2x^2+1}}\right)$	45 March 28, 2007
8	Find $f'(x)$ where $f(x)=\sqrt{x+1}\sin^3\left(\dfrac{x}{1+x}\right)$	2 May 20, 1993
9	Find $f'(x)$ where $f(x)=\dfrac{\tan(3x^2+5)}{1+\sin^2 x}$	
10	Find $f'(x)$ where $f(x)=\dfrac{\sec x}{x^3+\cot x}$	38 March 31, 2004
11	Find $f'(x)$ where $f(x)=\dfrac{x^2\sin x}{1+\cos x}$	41 March 30, 2005
12	Find $f'(x)$ where $f(x)=\sin^3(x^2+7)$	54 November 16, 2009 A

Homework

13	Find $f'(x)$ where $f(x)=\sin(\cos x)+\frac{\sec(3x)}{2x-1}$	10 June 6, 1994
14	Find $f'(x)$, where $f(x)=\sin^4(5x^3+x+2)-\csc\sqrt{x^2+1}$	40 October 28, 2004 A
15	If $f(x)=\sqrt[3]{3x-\sin^2(4x)}$, find $f'(x)$.	26 June 7, 2003
16	Find $f'(1)$ where $f(x)=\sqrt{1+x^2}-\frac{x^3}{x^4+1}$.	33 January 20, 2009
17	Find $f'(x)$ where $f(x)=\frac{x\cot x^2}{x^5+1}$	35 August 15, 2009
18	Let $f(x)=\sin(\cos x)+\sec(3x)$. Find $f'(x)$	
19	Find $f'(t)$ If $f(t)=\tan\left(\sqrt{t^2+1}\right)+t(t^2+1)^5$	26 May 10, 2001
20	Find y' if $y=\left[\tan\left(\frac{x}{2}\right)+\cos 2x\right]^3$	47 November 10, 2007 A
21	If $y=\sin^2\sqrt{5x^3-2x}+\tan\left(\frac{x+1}{x-2}\right)$. Find y'	14 March 28, 1996
22	Find $\frac{dy}{dx}$ where $y=\frac{(x^2+1)^3\tan x}{x}$	36 January 17, 2010
23	Find $\frac{dy}{dx}$ where $y=\sec(x)\tan(x)+\frac{6}{x^{\frac{2}{3}}}$	49 July 5, 2008
24	Find $\frac{dy}{dx}$ where $y=\sec^2(3x)+\frac{x}{x^2-1}$	50 November 17, 2008 A

Homework

25	Find $\frac{dy}{dx}$, if $y = \tan\sqrt{x^2+1} + \cos^5(x^3+5)$.	28 Dec 20, 2001
26	Find $\frac{dy}{dx}$ If $y = [\sin(x^2+10)]^2$.	29 July 25th, 2002
27	Find $\frac{dy}{dx}$ where $y = \sec\left(\frac{\sqrt{x}}{x^2+1}\right)$	42 May 5, 2008
28	Find $\frac{dy}{dx}$ where $y = x\left[\sin\left(\frac{x^2+1}{x+1}\right)\right]^3$	29 June 4. 2007
29	Find the derivative of $y = \tan^2\sqrt{1+\sec^2 x^3}$	15 February 12.1996
30	Find the derivative of $y = \sec^3\left(\frac{x+1}{\sqrt{x}}\right)$	14 January 6, 1996
31	Let f be function such that $f'(x) = \frac{1}{x}$ find $\frac{d}{dx}(f(x^2+1))$	
32	If $y = 3x^2 - 2\sqrt{x+1}$ and $x = t^3 + t^2 + 1$ find $\left.\frac{dy}{dt}\right\|_{t=1}$	9 January 8, 1994
33	If $y = u^2 - 5u + 1$ and $u = \frac{x-1}{x+1}$ Find $\frac{dy}{dx}$	37 July 12, 2003 A
34	Let $y = \sqrt{3u^2+5u+3}$ and $u = 2 - \sec^2 x - \cot x$ Find $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$	44 November 9, 2006

Homework

35	Let $y = r^2 - 6\sqrt{r+1}$, and $r = 8\tan\frac{\pi}{t}$. Find $\frac{dy}{dt}$ at $t = 4$	25 December 10, 2000
36	Let $y = \frac{1}{r^2} - 2r$ and $r = 1 + \tan\left(\frac{2\pi}{t-1}\right)$ Find $\frac{dy}{dt}$ at $t = 3$.	29 July 25th, 2002
37	If $y = \frac{(u-1)^2}{u^2+1}$ and $u = \sec^2 x + 1$, then Find $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$	34 July 22, 2004
38	If $y = \sqrt{u^2+u+7}$ and $u = x^2 + \frac{1}{\sqrt[3]{1+\tan x}}$ then Find $\frac{dy}{dx}$ at $x = 0$	17 January 8 ,1997
39	Let $y = \sqrt[3]{2t^2+t-9}$ and $t = \cot 2x + \csc^2 2x - 4$ Find $\frac{dy}{dx}$ at $x = \frac{\pi}{8}$	21 May 27. 2001
40	Use the chain rule to find $\frac{dy}{dt}$ at $t = \frac{\pi}{4}$, where $y = \sqrt[3]{u^2+u-1}$ and $u = \sec^2 t - \frac{2}{\sqrt{3+\tan t}}$	30 Jan. 12. 2008
41	Find $f'(x)$, if $f(x) = \sin^5(x^3+1)^4$	19 July 29, 2000

Homework

42	[3 pts.] Find the derivative of the following functions. $f(x) = x[x + (x + \cos^2 x)^3]$	58 7April 2011
43	[3 pts.] Find the derivative of the following functions. $g(t) = \sqrt{\frac{t}{t^2+4}}$	58 7April 2011
44	Find $f'(x)$, where $f(x) = \sin^2(1+\sqrt{x})$ (3 pts.)	57 November 8, 2010
45	(3 Points) Let $y = \frac{1}{2}\sqrt{4v^2 - 4v + 1}$ and $v = \cos(x^2) + \frac{1}{2-\tan x}$. Find at $x = 0$	49 July 24, 2010
44	Find $\frac{dy}{dx}$ where $y = (x^2-5)^3 \sin(x^4 - 3x + 1)$	38 January 15, 2011
45	Let $f(x) = \sec^4\left(\frac{2\sqrt{x}+1}{x^2+5}\right)$ find $f'(x)$	8 August 28, 1993

45 8 August 28, 1993	Let $f(x) = \sec^4\left(\frac{2\sqrt{x}+1}{x^2+5}\right)$ find $f'(x)$

Solution

$$f'(x) = 4\sec^3\left(\frac{2\sqrt{x}+1}{x^2+5}\right)\cdot \sec\left(\frac{2\sqrt{x}+1}{x^2+5}\right)\tan\left(\frac{2\sqrt{x}+1}{x^2+5}\right)\cdot \frac{(x^2+5)\cdot\frac{1}{\sqrt{x}} - (2\sqrt{x}+1)(2x)}{(x^2+5)^2}$$